Exercise 2.1.5

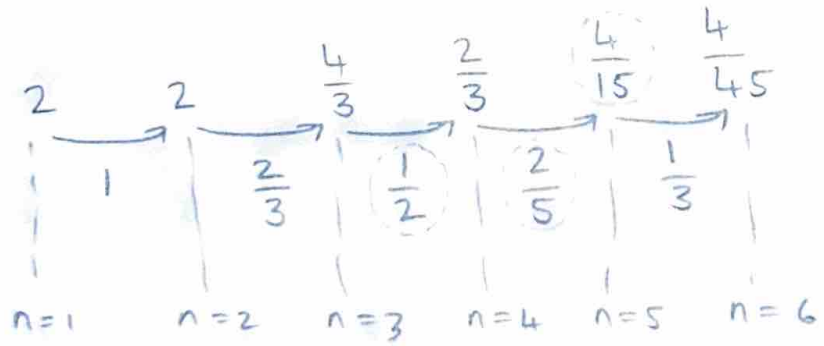
1a. $x_n = \dfrac{n(n+3)}{2n(2n+2)}$

$= \dfrac{n+3}{4n+4} \cdot \dfrac{\frac{1}{n}}{\frac{1}{n}}$ $\quad n \in \mathbb{N}$

$= \dfrac{1+\frac{3}{n}}{4+\frac{4}{n}}$

$\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \dfrac{1+\frac{3}{n}}{4+\frac{4}{n}} = \dfrac{1}{4}$

b. $x_n = \dfrac{2^n}{n!}$



We can see a pattern in the quotients:

$\frac{2}{2}, \frac{2}{3}, \frac{2}{4}, \frac{2}{5}, \frac{2}{6} \cdots \frac{2}{n+1}$

(5) The sequence becomes less than 1 at $n = 4$.

After $n = 4$, the nth element can be computed as:

$$\langle X \rangle_n |_{n>4} = \frac{2^4}{4!} \cdot \left(\frac{2}{5} \cdot \frac{2}{6} \cdots \frac{2}{n} \right)$$

which is bounded by $\left(\frac{2}{5}\right)^{n-4}$

If $\langle A \rangle_n \to L$ and $\langle C \rangle_n \to L$ and $\langle A \rangle_n \leq \langle D \rangle_n \leq \langle C \rangle_D$ then $\langle D \rangle_n \to L$ (squeeze theorem)

If we have series $\langle x \rangle_n = r^n$, $|r| < 1$, then

$\lim_{n \to \infty} r^n = 0$.

Thus $\lim_{n \to 0} \left(\frac{2}{5}\right)^{n-4} = 0$, then $\lim_{n \to 0} \frac{2^n}{n!} = 0$

(6) c. $x_n = 3 + (-1)^n$

$\langle x_n \rangle_n = 2, 4, 2, 4, \ldots$

Definition of convergence: $x_n \to L$ means for every $\varepsilon > 0$, eventually all items are within ε of L.

These terms do not occupy all $\varepsilon > 0$ from any L

d. $x_n = \sqrt{n+1} - \sqrt{n}$

$$x_n = \sqrt{n+1} - \sqrt{n} \cdot \frac{(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})}$$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$\lim_{n \to \infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$$

(7)

e. $x_n = \sqrt{n^2+n} - n$

$$x_n = \frac{\sqrt{n^2+n} - n}{1} \times \frac{\sqrt{n^2+n} + n}{\sqrt{n^2+n} + n}$$

$$= \frac{n^2 + n - n^2}{\sqrt{n^2+n} + n}$$

$$= \frac{n}{\sqrt{n^2+n} + n} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \frac{1}{\frac{n^2\sqrt{1+\frac{1}{n}}}{n} + 1}$$

$$= \frac{1}{\sqrt{1+\frac{1}{n}} + 1}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}} + 1} = \frac{1}{2}$$

(8)

2. I thaght the $\mathbb{Q}$ would be closed under limits, and could not feasably turn into irrationals, but I was wrong apparently.

In Exercise 2.1.4.1, we saw the digits of pi being generated by an algorithm using rationals, and if those digits were combined successively, the nth term would converge to π, as they became all $\varepsilon > 0$ values closer to π. So yes, rationals are not closed in this case.

Irrationals can converge to $\mathbb{Q}$:

$$\lim_{n \to \infty} \frac{\pi}{n} = 0.$$